J. Fluid Mech. (1987), *vol.* 179, *pp.* 371–381
Printed in Great Britain

Resonant reflection of water waves in a long channel with corrugated boundaries

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(Received 16 June 1986 and in revised form 21 October 1986)

A one-dimensional wave equation is derived for water-wave propagations in a long channel with corrugated boundaries. The amplitude and the wavelength of boundary undulations are assumed to be smaller than and in the same order of magnitude as the incident wavelength, respectively. When the Bragg reflection condition (i.e. the wavenumber of the boundary undulations is twice that of the incident wavenumber) is nearly satisfied, significant wave reflection could occur. Coupled equations for transmitted and reflected wave fields are derived for the near resonant coupling. The detuning mechanism is attributed to the slight deviation in the wavenumber of the corrugated boundaries from the Bragg wavenumber. Analytical solutions are obtained for the cases where the boundary undulations are within a finite region. The application of the present theory to the design of a harbour resonator is discussed.

1. Introduction

The interactions between water waves and a periodic seabed have been investigated by many researchers (e.g. Davies 1982; Davies & Heathershaw 1984; Mitra & Greenberg 1984; Mei 1985; Kirby 1986; Dalrymple & Kirby 1986). The most interesting feature of the interaction phenomena is that reflected waves can be resonated by the rippled seabed, if the wavelength of seabed undulations is one-half of that of the incident waves. This kind of resonant reflection is commonly known as Bragg reflection in crystallography (e.g. Yariv & Yeh 1984). In the case of a finite patch of periodic sandbars on a constant mean depth, Mei (1985) obtained analytical solutions near the resonance condition. In Mei's analysis, the wavelength of the sandbars is fixed and the uniform incident wave is detuned from the Bragg resonance condition by a small wavenumber K. Kirby (1986) derived a general two-dimensional wave equation which included the effects of slowly varying depth and rapidly varying, small-amplitude depth undulations. Numerical solutions were then obtained for one-dimensional problems concerning a finite patch of parallel periodic sandbars. In Kirby's numerical solutions, the wavenumber of the incident-wave train is fixed and the wavenumber of the sandbars is shifted from the resonance condition.

In this paper, we first derive a one-dimensional wave equation describing wave propagations in a long channel with corrugated boundaries. The boundary undulations are caused by either a rippled topography or the irregularities in channel banks. We investigate the reflected and transmitted waves near the resonant condition. The detuning mechanism in our analysis is caused by the small deviation of the wavenumber of the corrugated boundaries $\frac{1}{2}\Delta\beta$ from that of the Bragg reflection condition. Analytical solutions are obtained for the general case. In the case of constant channel width, owing to different detuning mechanisms, our solutions differ slightly from Mei's results when the resonance condition is not satisfied.

As a practical application, the phenomenon of the resonant reflection by a periodic undulation in channel width is examined as an alternative design of a harbour resonator (James 1970). The conventional harbour resonator consists of a rectangular branch canal on a narrow main channel. Quarter-wavelength resonance requires that the length of the branch canal is roughly one-quarter of the incident wavelength. In the present design, however, the amplitude of channel bank undulations is small in comparison with the wavelength. On the other hand, for the present design to be effective, the length of the undulation region must be in the order of magnitude of the channel width.

In the following sections, we derive first the one-dimensional wave equation. The coupled equations for transmitted and reflected wave fields are given and discussed in §3. In §4, analytical solutions are obtained for the case where the corrugated boundaries are confined in a finite region. A special application of the present theory to the design of a harbour resonator is presented in §5.

2. Theoretical derivation of quasi-one-dimensional wave equations

The depth-integrated two-dimensional wave equation, describing the propagation of a wave train over small-amplitude topographical undulations, was recently derived by Kirby (1986). Consider the x-axis as the primary wave propagation direction and the total still-water depth, $z=-h(x)$, as the sum of a constant $\bar{h}$ and a rapid undulation $\tilde{h}(x)$, i.e.

$$h(x)=\bar{h}+\tilde{h}(x). \tag{2.1}$$

The amplitude of the topographical undulations is small in comparison with the typical wavelength, $k\tilde{h} \ll 1$. The lengthscale of the topographical variation is, however, in the same order of magnitude as the incident wave wavelength,

$$O\left(\frac{1}{k\tilde{h}}\frac{\mathrm{d}\tilde{h}}{\mathrm{d}x}\right) \sim O(1).$$

If the leading-order velocity potential is expressed as

$$\Phi(x,y,z,t)=\frac{\cosh k(z+\bar{h})}{\cosh k\bar{h}}\,\phi(x,y,t), \tag{2.2}$$

with

$$\omega^2 = gk\tanh k\bar{h}, \tag{2.3}$$

where ω is the wave frequency and k the wavenumber, the wave equation can be written as (Kirby 1986):

$$\frac{\partial^2\phi}{\partial t^2}-\nabla\cdot\left[\left(CC_{\mathrm{g}}+\frac{g\tilde{h}}{\cosh^2 k\bar{h}}\right)\nabla\phi\right]+(\omega^2-k^2CC_{\mathrm{g}})\,\phi=0, \tag{2.4}$$

where $C=\omega/k$, and $C_{\mathrm{g}}=\mathrm{d}\omega/\mathrm{d}k$ are the phase velocity and the group velocity respectively. The gradient operator in (2.4) is a two-dimensional horizontal operator, i.e. $\nabla=(\partial/\partial x, \partial/\partial y)$. We remark here that in the original mild-slope equation derived by Kirby (1986) the mean depth is a slowly varying function; i.e. $O(\nabla\bar{h}/k\bar{h}) \ll 1$, and the coefficient $g/\cosh^2 k\bar{h}$ is outside of the divergent operator.

Consider wave propagations in a long channel. Let x be the longitudinal axis and $b(x)$ the width of the channel. If $y = a_1(x)$ and $a_2(x)$ describe the configuration of the channel banks, then $b(x) = a_2 - a_1$. The zero-flux boundary condition along the channel banks requires

$$\frac{\partial\phi}{\partial x}\frac{\partial a_{1,2}}{\partial x} - \frac{\partial\phi}{\partial y} = 0, \quad \text{on } y = a_{1,2}. \tag{2.5}$$

We now integrate (2.4) from $y = a_1$ to a_2 and apply Leibniz's rule to get

$$\frac{\partial^2}{\partial t^2}\int_{a_1}^{a_2}\phi\,\mathrm{d}y - \frac{\partial}{\partial x}\int_{a_1}^{a_2}\left[CC_g + \frac{g\tilde{h}}{\cosh^2 k\bar{h}}\right]\frac{\partial\phi}{\partial x}\,\mathrm{d}y - \left(CC_g + \frac{g\tilde{h}}{\cosh^2 k\bar{h}}\right)\left[\frac{\partial\phi}{\partial x}\frac{\partial a_1}{\partial x} - \frac{\partial\phi}{\partial y}\right]\Bigg|_{y=a_1}$$
$$+\left(CC_g + \frac{g\tilde{h}}{\cosh^2 k\bar{h}}\right)\left[\frac{\partial\phi}{\partial x}\frac{\partial a_2}{\partial x} - \frac{\partial\phi}{\partial y}\right]\Bigg|_{y=a_2} + (\omega^2 - k^2CC_g)\int_{a_1}^{a_2}\phi\,\mathrm{d}y = 0.$$

If the channel width is small compared with the characteristic wavelength, $kb \ll 1$, the lateral variation in the wave field becomes less important. Employing the boundary conditions (2.5) along the channel banks and assuming that the variation of ϕ in the y-direction can be ignored, we reduce the above equation to

$$\frac{\partial^2\phi}{\partial t^2} - \frac{1}{b}\frac{\partial}{\partial x}\left[A\frac{\partial\phi}{\partial x}\right] + (\omega^2 - k^2CC_g)\,\phi = 0, \tag{2.6}$$

where
$$A = \left(CC_g + \frac{g\tilde{h}}{\cosh^2 k\bar{h}}\right)b. \tag{2.7}$$

Equations (2.6) and (2.7) represent the quasi-one-dimensional wave equation in a long channel with width $b(x)$. In the case where the wave field is periodic in time, i.e.

$$\phi = \Phi(x)\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{2.8}$$

the wave equation, (2.6), can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[A\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right] + k^2CC_g\,b\Phi = 0. \tag{2.9}$$

The wave transformation is caused by the variation of channel width b and the topographical undulations $\tilde{h}$.

Introducing the transformation

$$\xi = \phi A^{\frac{1}{2}} \tag{2.10}$$

into the general wave equation, (2.6), we obtain

$$\frac{b}{A}\left[\frac{\partial^2\xi}{\partial t^2} + \omega^2\xi\right] - \frac{\partial^2\xi}{\partial x^2} - k^2\left[\frac{CC_g b}{A} - \frac{1}{2k^2A^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{A^{\frac{1}{2}}}\frac{\mathrm{d}A}{\mathrm{d}x}\right)\right]\xi = 0. \tag{2.11}$$

We now assume that the channel width can also be decomposed into two parts: $\bar{b}$ is a slowly varying function of x, but $\tilde{b}$ represents fast undulations in channel width. The amplitude of the width undulations is also assumed to be small, i.e.

$$O\left(\frac{1}{k\bar{b}}\frac{\mathrm{d}\bar{b}}{\mathrm{d}x}\right) \approx O(k\tilde{b}) \approx O(k\tilde{h}) \ll 1; \quad O\left(\frac{1}{k\tilde{b}}\frac{\mathrm{d}\tilde{b}}{\mathrm{d}x}\right) \approx O(1). \tag{2.12}$$

Substituting $b = \bar{b}+\tilde{b}$ into (2.11) and neglecting terms smaller than $O(k\tilde{b})$, we obtain

$$\frac{b}{A}\left[\frac{\partial^2\xi}{\partial t^2}+\omega^2\xi\right]-\frac{\partial^2\xi}{\partial x^2}-\bar{k}^2\xi = 0, \tag{2.13}$$

where

$$\bar{k}^2 = k^2\left[1-\frac{g\tilde{h}}{CC_g\cosh^2 k\bar{h}}-\frac{g}{2k^2CC_g\cosh^2 k\bar{h}}\frac{d^2\tilde{h}}{dx^2}-\frac{1}{2k^2\bar{b}}\frac{d^2\tilde{b}}{dx^2}\right], \tag{2.14}$$

and

$$\frac{b}{A} = \frac{1}{CC_g}\left[1-\frac{g\tilde{h}}{CC_g\cosh^2 k\bar{h}}\right]. \tag{2.15}$$

The effects of water depth and channel width undulations are separated. For the wave field which is periodic in time, i.e.

$$\xi = \zeta\, e^{-i\omega t}, \tag{2.16}$$

(2.13) reduces to the one-dimensional Helmholtz equation

$$\frac{d^2\zeta}{dx^2}+\bar{k}^2\zeta = 0, \tag{2.17}$$

with the variable refraction index $\bar{k}$ given in (2.14).

3. Resonant reflection

Due to the undulations in water depth and channel width, significant wave reflection could occur under the resonance condition (Davies & Heathershaw 1984). The total wave field consists of incident waves and reflected waves; i.e.

$$\xi = \tfrac{1}{2}\Psi^+(x,t)\, e^{i(kx-\omega t)}+\tfrac{1}{2}\Psi^-(x,t)\, e^{-i(kx+\omega t)}+\text{C.C.}, \tag{3.1}$$

where Ψ^+ and Ψ^-, representing the envelopes of incident and reflected wave amplitudes, respectively, are assumed to be slowly varying functions in both x and t. Thus, we approximate the first derivatives of ξ with respect to x and t to be

$$\frac{\partial\xi}{\partial x} \approx \tfrac{1}{2}ik[\Psi^+\, e^{i(kx-\omega t)}-\Psi^-\, e^{-i(kx+\omega t)}]+\text{C.C.}, \tag{3.2}$$

$$\frac{\partial\xi}{\partial t} \approx -\tfrac{1}{2}i\omega[\Psi^+\, e^{i(kx-\omega t)}+\Psi^-\, e^{-i(kx+\omega t)}]+\text{C.C.} \tag{3.3}$$

Substituting (3.1), (3.2) and (3.3) into (2.13), we obtain two equations for Ψ^+ and Ψ^-:

$$\frac{\partial\Psi^+}{\partial t}+C_g\frac{\partial\Psi^+}{\partial x} = \frac{iCC_g}{2\omega}k^2\Delta\{\Psi^+ + \Psi^-\, e^{-2ikx}\}-\frac{\partial\Psi^-}{\partial t}e^{-2ikx}, \tag{3.4}$$

$$\frac{\partial\Psi^-}{\partial t}-C_g\frac{\partial\Psi^-}{\partial x} = \frac{iCC_g}{2\omega}k^2\Delta\{\Psi^- + \Psi^+\, e^{2ikx}\}+\frac{\partial\Psi^+}{\partial t}e^{2ikx}, \tag{3.5}$$

where

$$\Delta = \frac{\bar{k}^2}{k^2}-1 = -\left\{\frac{g\tilde{h}}{CC_g\cosh^2 k\bar{h}}+\frac{g}{2k^2CC_g\cosh^2 k\bar{h}}\frac{d^2\tilde{h}}{dx^2}+\frac{1}{2k^2\bar{b}}\frac{d^2\tilde{b}}{dx^2}\right\}. \tag{3.6}$$

Since both $\tilde{h}$ and $\tilde{b}$ are periodic in x, we can expand Δ as a Fourier series

$$\Delta = \sum_{m\neq 0}\Delta_m \exp[-imAx], \quad m = \pm 1, \pm 2, \ldots, \tag{3.7}$$

where Δ_{-m} is the complex conjugate of Δ_m.

Substitution of (3.7) into (3.4) and (3.5) leads to

$$\frac{\partial \Psi^+}{\partial t}+C_g\frac{\partial \Psi^+}{\partial x}=\frac{\mathrm{i}CC_g}{2\omega}k^2\sum_{m\neq 0}\Delta_m\{\Psi^+\,\mathrm{e}^{-\mathrm{i}m\Lambda x}+\Psi^-\,\mathrm{e}^{-\mathrm{i}(2k+m\Lambda)x}\}-\frac{\partial \Psi^-}{\partial t}\mathrm{e}^{-2\mathrm{i}kx}, \quad (3.8)$$

$$\frac{\partial \Psi^-}{\partial t}-C_g\frac{\partial \Psi^-}{\partial x}=\frac{\mathrm{i}CC_g}{2\omega}k^2\sum_{m\neq 0}\Delta_m\{\Psi^-\,\mathrm{e}^{-\mathrm{i}m\Lambda x}+\Psi^+\,\mathrm{e}^{\mathrm{i}(2k-m\Lambda)x}\}+\frac{\partial \Psi^+}{\partial t}\mathrm{e}^{2\mathrm{i}kx}. \quad (3.9)$$

Equations (3.8) and (3.9) constitute a set of coupled differential equations for the incident and reflected wave fields. The right-hand-side terms in (3.8) and (3.9) contain the fast oscillating terms and a slowly varying term where m satisfies the condition

$$m\Lambda=\mp 2k \quad (3.10)$$

for (3.8) and (3.9) respectively. Equation (3.10) is also known as the Bragg reflection condition. Since the incident and reflected wave envelopes, Ψ^+ and Ψ^-, are slowly varying functions of x, in the integration of (3.8) and (3.9) the only significant contribution comes from the term (nearly) satisfying the Bragg reflection condition. Thus, near the resonant coupling, (3.8) and (3.9) can be simplified to

$$\frac{\partial \Psi^+}{\partial t}+C_g\frac{\partial \Psi^+}{\partial x}=\frac{\mathrm{i}CC_g}{2\omega}k^2\Delta_{-m}\,\Psi^-\,\mathrm{e}^{-\mathrm{i}\Delta\beta x}, \quad (3.11)$$

$$\frac{\partial \Psi^-}{\partial t}-C_g\frac{\partial \Psi^-}{\partial x}=\frac{\mathrm{i}CC_g}{2\omega}k^2\Delta_{m}\,\Psi^+\,\mathrm{e}^{\mathrm{i}\Delta\beta x}, \quad (3.12)$$

where
$$\Delta\beta=2k-m\Lambda\ll 1, \quad (3.13)$$

and $\frac{1}{2}\Delta\beta$ is the detuning wavenumber of the corrugated boundaries. The coupling between incident and reflected waves is strong as long as the Bragg condition is nearly satisfied and the coupling coefficient $\Delta_{\pm m}$ is not negligible. The coupling coefficient $\Delta_{\pm m}$ can be specifically defined as follows. If the topographical and channel width undulations are expressed as

$$\tilde{h}=\tfrac{1}{2}\sum_{m\neq 0}D_m\,\mathrm{e}^{-\mathrm{i}m\Lambda x}, \quad (3.14a)$$

$$\tilde{b}=\tfrac{1}{2}\sum_{m\neq 0}B_m\,\mathrm{e}^{-\mathrm{i}m\Lambda x}, \quad (3.14b)$$

we obtain, from (3.6) and (3.7), that

$$\Delta_m=-\frac{gD_m}{2CC_g\,k^2\cosh^2 k\bar{h}}[k^2-\tfrac{1}{2}(m\Lambda)^2]+\frac{B_m}{k^2\bar{b}}(m\Lambda)^2. \quad (3.15)$$

Near the resonant coupling, the coupling coefficient becomes

$$\Delta_m=\frac{gD_m}{2CC_g\cosh^2 k\bar{h}}\left[1-\frac{2\Delta\beta}{k}+\frac{(\Delta\beta)^2}{k^2}\right]+\frac{B_m}{\bar{b}}\left[1-\frac{\Delta\beta}{k}+\frac{(\Delta\beta)^2}{4k^2}\right]. \quad (3.16)$$

Note that the coupling coefficient is, in general, a function of the detuning wavenumber $\frac{1}{2}\Delta\beta$. We remark here that if the channel width is uniform, $B_m=0$, and if $\Delta\beta=0$, (3.11) and (3.12) reduce to those derived by Mei (1985).

4. Analytical solutions in a channel with a finite patch of corrugated boundaries

In this section we consider the problem where the channel width and topographical undulations are confined within a finite distance, $0 < x < L$. The uniform incident wave train, arriving from $x \sim -\infty$, has a wavenumber k, and a constant amplitude. Thus

$$\Psi^+ = \Psi_0, \quad x < 0, \tag{4.1}$$

and Ψ_0 is a prescribed constant. Equation (4.1) satisfies the governing equation (3.11) in the incidence region $x < 0$. We impose that there be no reflected waves in the region $x > L$; i.e.

$$\Psi^- = 0, \quad x > L. \tag{4.2}$$

The governing equations for $x < 0$ and $x > L$ become

$$\frac{\mathrm{d}\Psi^+}{\mathrm{d}x} = 0, \quad x > L; \quad \frac{\mathrm{d}\Psi^-}{\mathrm{d}x} = 0, \quad x < 0, \tag{4.3}$$

Over the undulations region, $0 < x < L$, the governing equations are given in (3.11) and (3.12). Continuity of $\Psi^\pm$ at $x = 0$ and L gives four conditions. The solution in all three regions can be readily found.

Over the undulations region, $0 < x < L$, the incident and reflected wave fields are expressed as

$$\Psi^+ = \Psi_0 T(x), \quad 0 < x < L, \tag{4.4}$$

$$\Psi^- = \Psi_0 R(x), \quad 0 < x < L, \tag{4.5}$$

where

$$T(x) = \mathrm{e}^{-\mathrm{i}(\frac{1}{2}\Delta\beta)x}\left\{\frac{s\cosh s(L-x) - \mathrm{i}(\frac{1}{2}\Delta\beta)\sinh s(L-x)}{s\cosh sL - \mathrm{i}(\frac{1}{2}\Delta\beta)\sinh sL}\right\}, \tag{4.6}$$

$$R(x) = \mathrm{e}^{\mathrm{i}(\frac{1}{2}\Delta\beta)x}\left\{\frac{\mathrm{i}\,k\Delta_m \sinh s(L-x)}{2s\cosh sL - 2\mathrm{i}(\frac{1}{2}\Delta\beta)\sinh sL}\right\}, \tag{4.7}$$

and the wavenumber s is the solution of

$$s^2 = \tfrac{1}{4}k^2|\Delta_m|^2 - (\tfrac{1}{2}\Delta\beta)^2. \tag{4.8}$$

The wavenumber s could be real or imaginary depending on the sign of the right-hand side of (4.8). If s is imaginary, the hyperbolic functions in (4.6) and (4.7) become trigonometric functions. The cut-off frequency Ω_0 is defined as the frequency corresponding to $s = 0$. On the incident side, $x < 0$, the reflected waves are

$$\Psi^- = \Psi_0 R(0), \quad x < 0. \tag{4.9}$$

The reflection coefficient $|R(0)|$ can be found from (4.7). Thus

$$|R(0)| = \left|\frac{\mathrm{i}\,k\Delta_m \sinh sL}{2[s\cosh sL - \mathrm{i}(\frac{1}{2}\Delta\beta)\sinh sL]}\right|. \tag{4.10}$$

On the transmission side, $x > L$, the transmitted waves are

$$\Psi^+ = \Psi_0 T(L), \quad x > L, \tag{4.11}$$

where the transmission coefficient $|T(L)|$ can be obtained from (4.6) as

$$|T(L)| = \left|\frac{s}{s\cosh sL - \mathrm{i}(\frac{1}{2}\Delta\beta)\sinh sL}\right|. \tag{4.12}$$

From (4.10) and (4.12), the wave energy is conserved; i.e. $|R(0)|^2+|T(L)|^2 = 1$. The reflected wave intensity over the undulated region is

$$\frac{\rho\omega^2}{2g}\left|\frac{\Psi^-}{A^{\frac{1}{2}}}\right|^2 = \frac{\rho\omega^2}{2g}\left|\frac{\Psi_0}{A^{\frac{1}{2}}}R(x)\right|^2, \quad 0 < x < L, \tag{4.13}$$

where $R(x)$ and A are given in (4.7) and (2.7) respectively. Since A is a function of $\bar{h}$ and $\bar{b}$, the reflected wave intensity responds to the undulations.

In the case of perfect tuning, $\Delta\beta = 0$, the transmitted and reflected wave fields become

$$T(x) = \frac{\cosh[\frac{1}{2}k|\varDelta_m|(L-x)]}{\cosh[\frac{1}{2}k|\varDelta_m|L]}, \tag{4.14}$$

$$R(x) = \frac{\mathrm{i}\sinh[\frac{1}{2}k|\varDelta_m|(L-x)]}{\cosh[\frac{1}{2}k|\varDelta_m|L]}\frac{\varDelta_m}{|\varDelta_m|}, \tag{4.15}$$

$$\varDelta_m = \frac{gD_m}{2CC_g\cosh^2 k\bar{h}}+\frac{4B_m}{\bar{b}}. \tag{4.16}$$

The maximum reflection coefficient is given as

$$R_{\max} = |R(0)| = \tanh[\tfrac{1}{2}k|\varDelta_m|L] \tag{4.17}$$

The reflection coefficient could become zero when $\varDelta_m$ vanishes. From (3.14) and (4.16), $\varDelta_m$ will become zero only when D_m and B_m have opposite signs; the topographical and channel width undulations must be 180° out of phase.

In Mei's (1985) analysis, the uniform incident wave train is detuned by a small wavenumber K with an associated detuning frequency $\varOmega$. His solutions can be converted into our solutions with a substitution $\frac{1}{2}\Delta\beta = K$. Slight differences between Mei's and the present solution, however, appear in the coupling coefficient, (3.16), as well as in the determination of the cut-off frequency. For the coupling coefficient, our solution (3.16), contains terms in the order of magnitude of $\frac{1}{2}\Delta\beta$. For the case where $B_m = 0$, the cut-off frequency $\varOmega_0$ can be obtained by setting (4.8) zero to get $\varOmega_0 = \frac{1}{2}kC_g\varDelta_m = gk^2D_m/4\omega\cosh^2 k\bar{h}$, which was also obtained by Mei (1985). We reiterate here that in the general situation, $\Delta\beta \neq 0$, the coupling coefficient $\varDelta_m$ is a function of $\Delta\beta$. The cut-off frequency is quite different from Ω_0. To illustrate this, we rewrite (4.8) in the following form:

$$\left(\frac{sC_g}{\varOmega_0}\right)^2 = \left[1-8\left(\frac{\Delta\beta}{2k}\right)+24\left(\frac{\Delta\beta}{2k}\right)^2\right]-\left(\frac{\Delta\beta}{2k}\right)^2\left(\frac{kC_g}{\varOmega_0}\right)^2. \tag{4.18}$$

In figure 1 we plot $(sC_g/\varOmega_0)^2$ *vs.* $(\Delta\beta/2k)^2(kC_g/\varOmega_0)^2$ for different values of $\varOmega_0/kC_g$. In Mei's (1985) analysis $\Delta\beta$ does not appear in $\varDelta_m$, the equation equivalent to (4.18) becomes

$$\left(\frac{sC_g}{\varOmega_0}\right)^2 = 1-\left(\frac{\Delta\beta}{2k}\right)^2\left(\frac{kC_g}{\varOmega_0}\right)^2. \tag{4.19}$$

The cut-off frequency is obtained when (4.19) is zero.

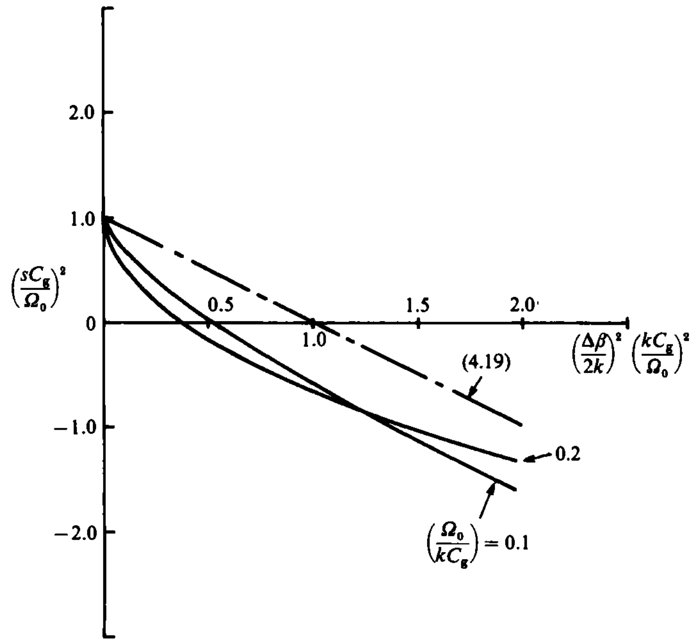

FIGURE 1. The sign of the square of the wavenumber s as a function of $\Delta\beta$ and Ω_0.

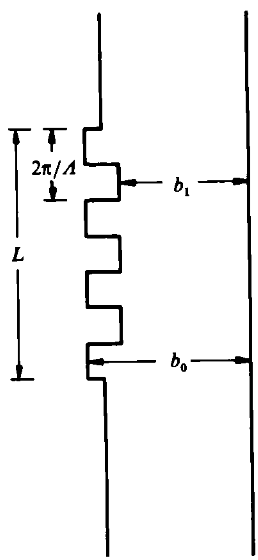

FIGURE 2. Definition sketch of a harbour resonator.

5. A harbour resonator

In this section we examine the possible utilization of the concept of Bragg reflection for the design of a harbour resonator. The goal of a harbour resonator is to reflect incident waves from the harbour entrance channel so that the wave motion in the harbour will be small.

Consider a simple case where the water depth in the channel is a constant; i.e. $\tilde{h} = 0$, and the channel width is described as (see figure 2)

$$b(x) = \tfrac{1}{2}(b_0+b_1)+\tfrac{1}{2}(b_0-b_1)f(x), \quad 0 < x < L, \tag{5.1}$$

where

$$f(x) = \begin{cases} 1, & \dfrac{n\pi}{\Lambda} < x < \dfrac{(n+1)\pi}{\Lambda}, \\ -1, & \dfrac{(n+1)\pi}{\Lambda} < x < \dfrac{(n+2)\pi}{\Lambda}. \end{cases} \tag{5.2}$$

By using the Fourier series for $f(x)$, the channel width can be written as

$$b(x) = \bar{b}+\tilde{b}, \tag{5.3a}$$

$$\bar{b} = \tfrac{1}{2}(b_0+b_1), \tag{5.3b}$$

$$\tilde{b} = \tfrac{1}{2}\Sigma B_m \,\mathrm{e}^{-\mathrm{i}m\Lambda x}, \tag{5.3c}$$

$$B_m = (b_0-b_1)\frac{\mathrm{i}(1-\cos m\pi)}{m\pi} = \begin{cases} 0, & m = \text{even}, \\ \dfrac{2i(b_0-b_1)}{m\pi}, & m = \text{odd}. \end{cases} \tag{5.3d}$$

The corresponding coupling coefficient is

$$\Delta_m = \frac{4\mathrm{i}(b_0-b_1)}{m\pi(b_0+b_1)}\left[1-\frac{\Delta\beta}{k}+\frac{(\Delta\beta)^2}{4k^2}\right]. \tag{5.4}$$

The wavenumber s is simply defined as

$$s^2 = \frac{4k^2(b_0-b_1)^2}{m^2\pi^2(b_0+b_1)^2}\left[1-\frac{\Delta\beta}{k}+\frac{(\Delta\beta)^2}{4k^2}\right]^2-\frac{k^2}{4}\left(\frac{\Delta\beta}{k}\right)^2. \tag{5.5}$$

The transmitted and reflected wave fields can be deduced from (4.6) and (4.7). Near the resonance the reflection coefficient $|R(0)|$ can be approximated as

$$|R(0)| = \frac{2k(b_0-b_1)\sinh sL}{m\pi[s\cosh sL-\mathrm{i}(\tfrac{1}{2}\Delta\beta)\sinh sL](b_0+b_1)}. \tag{5.6}$$

The maximum reflection coefficient can be found from (4.17). Thus

$$R_{\max} = \tanh\left[\frac{2k(b_0-b_1)}{m\pi(b_0+b_1)}L\right]. \tag{5.7}$$

Therefore, the reflection coefficient approaches one when the parameter

$$\kappa = \frac{2k(b_0-b_1)}{m\pi(b_0+b_1)}L \tag{5.8}$$

becomes very large, which can be achieved if the undulation region is long in comparison with the mean channel width $\bar{b}$. On the other hand, the reflection coefficient becomes rather small when κ is small. This could happen when $O(L/\bar{b}) < O(1)$ or $m = 2k/\Lambda$ is large (i.e. the wavelength of the boundary undulations

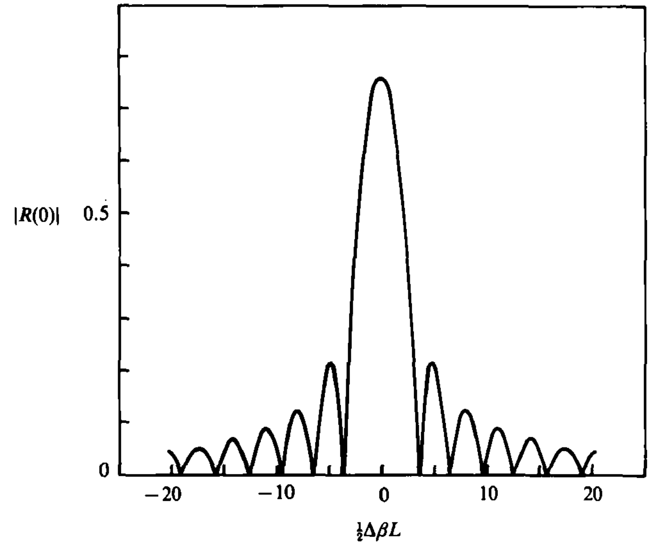

FIGURE 3. Reflection coefficient for $\kappa = 1.0$.

is much longer than the incident wavelength). In the case where $(b_0 - b_1)/\lambda = 0.1(\lambda = 2\pi/k)$, $m = 3$, and $L/\bar{b} = 15$, the parameter κ is 1.0. The corresponding reflection coefficient $|R(0)|$ is presented in figure 3. The maximum reflection is 0.762. Because the amplitude of the bank undulations is small compared to the incident wavelength, the present approach is an attractive alternative to the conventional quarter-wavelength harbour resonator.

6. Concluding remarks

A general one-dimensional wave equation describing wave propagation in a long channel with corrugated boundaries is derived. The boundary variations consist of slow varying and fast varying, but small-amplitude components. The sources for the boundary undulations include the topographical changes (e.g. sandbars) and the irregularities in the channel width. Analytical solutions are obtained for the transmitted and reflected waves over a finite length of the undulation region near the Bragg reflection condition. The concept of the resonance reflection is suggested for use as an alternative design for a harbour resonator. Experimental verification of the present theory is underway.

The present theory is limited to small-amplitude waves in an intermediate water depth. It can, however, be extended to the shallow-water limit and include nonlinear effects.

This research was supported by the New York Sea Grant Institute. The discussions with Dr Chiang C. Mei and Mr Sung B. Yoon have been very fruitful.

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